

The connection between statics and dynamics of spin glasses

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We present results of numerical simulations on a one-dimensional Ising spin glass with long-range interactions. Parameters of the model are chosen such that it is a proxy for a short-range spin glass *above* the upper critical dimension (i.e. in the mean-field regime). The system is quenched to a temperature well below the transition temperature T_c and the growth of correlations is observed. The spatial decay of the correlations at distances less than the dynamic correlation length $\xi(t)$ agrees quantitatively with the predictions of a *static* theory, the “metastate”, evaluated according to the replica symmetry breaking (RSB) theory. We also compute the dynamic exponent $z(T)$ defined by $\xi(t) \propto t^{1/z(T)}$ and find that it is compatible with the mean-field value of the *critical* dynamical exponent for short range spin glasses.

Experimental measurements on a system at finite-temperature involve a *time average*. Dynamics is harder to calculate than statics, so, in theoretical work, the time average is usually replaced by a static calculation using statistical mechanics in which one *sums over all configurations* with the Boltzmann probability distribution. Most systems are ergodic, so theory agrees with experiment even though different averages are performed. One situation where more care is needed is that of a phase transition where symmetry is spontaneously broken. A simple case is the Ising ferromagnet, which has just two ordered states below the transition temperature T_c , the “up” spin state with a net positive magnetization, and the “down” state. On cooling the system will choose one of these symmetry-related states and acquire a non-zero magnetization. However, the Boltzmann sum will (unphysically) include both the up and down states and give zero net magnetization. The “up” and “down” states in the Ising ferromagnet are called “pure states” in the literature [1]. In pure states, correlation functions have a “clustering” property which means that “connected” correlation vanish at infinity, i.e.

$$\lim_{|\mathbf{r}_i - \mathbf{r}_j| \rightarrow \infty} (\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle) = 0. \quad (1)$$

This does not occur in the Boltzmann average since the second term is zero, so the combination of “up” and “down” is not a pure state; rather it is a “mixed state”.

However, in spin glasses, which have disorder and “frustration”, the situation is much more complicated. *Dynamically*, below the spin glass transition temperature T_c a macroscopic system is not in thermal equilibrium because relaxation times are much too long. Rather, in a typical experiment the system is quenched from a high temperature to a temperature below T_c and the subsequent dynamic evolution of the system is observed. *For statics*, the state (or states) of thermal equilibrium are very complicated and are not related to any symmetry. As for the ferromagnet we would like to find a static calculation which will predict the experimental behavior, at least to some extent. In this paper we show *quantitatively* that the theoretical idea called the “metastate” [2, 3], combined with the technique of “replica symmetry breaking” (RSB) [4, 5] provides such a description for spin

glasses, at least in high dimensions, where the critical behavior is described by mean field theory.

Since the clustering property in Eq. (1) is convenient one would like to also describe spin glasses in terms of pure states. This can be done (in principle) by taking a very large system, applying some boundary condition on it, and studying the correlations in a relatively small window far from the boundary [1, 6]. This procedure is repeated for many different boundary conditions.

The question of whether there are many pure states or just one (a time-reversed pair in the absence of a magnetic field) in spin glasses has been very controversial [4, 5, 7, 8]. If there are many, one needs to do some sort of statistical average over them, which is called a “metastate”, for which different but equivalent formulations have been given by Newman and Stein [2] and by Aizenman and Wehr (AW) [3]. In the AW metastate, one considers a scale M , intermediate between the window size W and the system size L . The metastate-averaged state (MAS) is obtained by computing correlation functions in the window in which an average is performed not only over the spins but also over the bonds in the “exterior” region between M and L [3, 6, 9].

For the infinite-range model of Sherrington and Kirkpatrick (SK) [10], Parisi’s exact solution [4, 5], solved by RSB, has many pure states in a sense that was clarified later by Newman and Stein [2], see also Read [6].

The critical behavior of a realistic spin glass is expected to be the same as that of the SK [10] model in dimension d greater than the “upper critical dimension”, d_u , which is equal to six. However, this does not necessarily mean that the RSB description of the spin glass phase *below* T_c also applies for $d > 6$ [2, 7, 8]. Nonetheless Read [6] has computed the spatial fluctuations in a finite-dimensional model below T_c , assuming mean-field (Gaussian) fluctuations, and the metastate description coming from Parisi’s [4, 5] RSB solution of the SK model. Spin correlations are found [6, 11, 12] to decay with a power of the distance, due to the averaging over many pure states (which are unrelated by symmetry) in the metastate, i.e.

$$\langle S_i S_j \rangle_{\text{MAS}}^2 \propto 1/r_{ij}^{\alpha_s} \quad \text{with} \quad \alpha_s = d - 4, \quad (2)$$

where “s” refers to “static”, “MAS” stands for metastate-

averaged state, and sites i and j are in the window far from the boundary. The result in Eq. (2) had been obtained earlier in Ref. [12] from an RSB calculation working in the zero overlap sector.

We emphasize that the calculation leading to Eq. (2) is a *static* one. Is it possible to relate it to experiments (or numerical simulations), which concern (non-equilibrium) *dynamics*? Many simulations [9, 13–17] have been carried out in which a spin glass is quenched to below T_c and the resulting dynamics analyzed. It is found that fluctuations reach a steady state on length scales smaller than a dynamic correlation length $\xi(t)$ which is found, empirically, to grow with a power of t like

$$\xi(t) \propto t^{1/z(T)}, \quad (3)$$

where the non-equilibrium dynamic exponent $z(T)$ is found to vary, roughly, like $1/T$ and becomes close to the critical dynamical exponent, z_c , for $T = T_c$, [9, 13–19]

$$1/z(T) \simeq (T/T_c) z_c. \quad (4)$$

At distances less than $\xi(t)$ correlations are observed to fall off with a power of distance leading to the following scaling prediction

$$C_4(r_{ij}, t) \equiv [\langle S_i(t) S_j(t) \rangle^2] = \frac{1}{r_{ij}^{\alpha_d}} f\left(\frac{r_{ij}}{\xi(t)}\right), \quad (5)$$

where “d” refers to “dynamic”. Here the thermal average squared, $\langle \dots \rangle^2$, is performed by simulating two copies of the system with the same interactions, initialized with different random spin configurations. Spin configurations of the two copies at the same time are used in Eq. (5). Use of two copies provides an unbiased estimate of this thermal average. The second average, $[\dots]$, is over the bonds. We will also average over all pairs of sites a given distance r apart. For $r_{ij} \ll \xi(t)$ one finds $f(x \rightarrow 0) = \text{const.}$ so

$$C_4(r_{ij}, t) \propto 1/r_{ij}^{\alpha_d} \quad (r_{ij} \ll \xi(t)). \quad (6)$$

Clearly, the non-equilibrium dynamics is generating a sampling of the pure states. To our knowledge, White and Fisher [20] were the first to point out the similarity of this sampling and the metastate average for statics. They use the term “maturation metastate” to describe the ensemble of states generated dynamically on scales less than $\xi(t)$ following a quench, and “equilibrium metastate” for the static metastate discussed earlier. Here we will use terms “dynamic” and “static” to describe these two metastates. Subsequently Manssen et al. [21] emphasized the similarity between the two metastates and suggested that they might actually be equivalent, in which case α_s in Eq. (2) would equal α_d in Eq. (6). The rationale behind this hypothesis is that thermal fluctuations of the spins outside the window at a distance $\xi(t)$ and greater, which are not equilibrated with respect to spins in the window, effectively generate a random noise to the spins in the window which is

similar to the random perturbation coming from changing the bonds in the outer region according to the AW metastate. It would also be very *useful* if the metastates were equivalent because then a theory of the *statics* of spin glasses would give results corresponding to experiments, which are a *time average*, as is the case for simpler systems with a phase transition like ferromagnets.

For the three-dimensional spin glass, Refs. [22] and [23] have shown that an equilibrium calculation in the zero spin-overlap sector gives a power-law decay for the spin correlations, as in Eq. (2), with a value of α_s consistent with that obtained from dynamics following a quench in Ref. [17]. These are both numerical results. Here we wish to consider the mean field limit, $d > 6$, because there is an exact *analytic* result $\alpha_s = d - 4$, in RSB theory [6, 12] to compare with. Unfortunately it is difficult to carry out useful Monte Carlo simulations for a spin glass below T_c in more than six dimensions, because the number of sites in a region of linear size ℓ , increases so fast, as ℓ^d , that the range of ℓ that can be studied is very limited. The only calculation of the exponent α_d in the mean field region that we are aware of is that of Ref. [24] who studied $d = 6$ but only *at* $T = T_c$ and so these results are for the critical point rather than the spin glass phase.

However, it has been pointed out that a class of models in one-dimension, with long-range interactions falling off with a power of distance can serve as a proxy for short-range models [25–27] in a range of dimensions including high dimension. The Hamiltonian is

$$\mathcal{H} = - \sum_{i,j} J_{ij} S_i S_j, \quad (7)$$

where the sites $i = 1, 2, \dots, N$ lie on a one-dimensional chain with periodic boundary conditions, the Ising spins S_i take values ± 1 , and the interactions J_{ij} are independent random variables with whose distribution has mean and variance given by

$$[J_{ij}] = 0, \quad [J_{ij}^2] \propto 1/R_{ij}^{2\sigma}, \quad (8)$$

in which σ is a parameter which can be varied. To incorporate periodic boundary conditions it is convenient to put the sites on a ring and define R_{ij} to be the chord distance between i and j , i.e. $R_{ij} = (N/\pi) \sin(\pi|i-j|/N)$, whereas the distance *along the ring* is $r_{ij} = |i-j|$ if $|i-j| < N/2$ and $r_{ij} = N - |i-j|$ otherwise. The bonds are generated, and the constant of proportionality in Eq. (8) fixed, in the following way due to Ref. [28]. We choose an average coordination number z_b , which we take here to be $z_b = 6$. We choose a site i at random and then a site j with a probability $C/R_{ij}^{2\sigma}$, where C is the normalization constant. If there is already an interaction between i and j repeat until a pair (i, j) is found which has not occurred before. Then assign an interaction between i and j chosen from a Gaussian distribution with mean zero and standard deviation unity. Repeat this $Nz_b/2$ times, so there are $Nz_b/2$ interactions.

Varying σ is argued to be analogous to changing the dimension d of a short-range model [26]. In the mean field

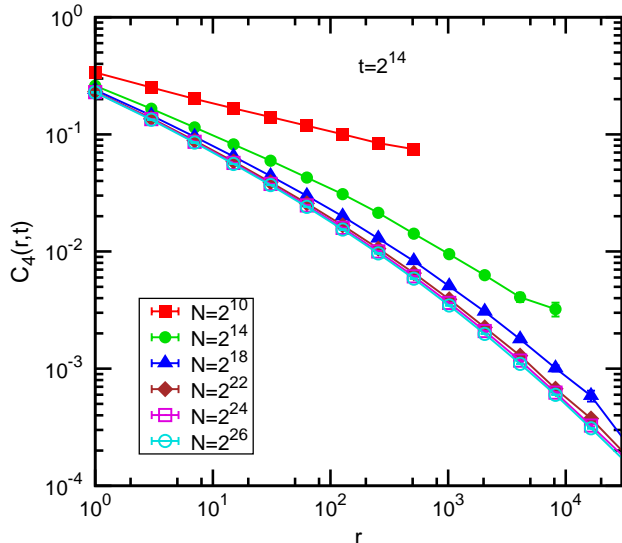


FIG. 1: Data for the correlation function $C_4(r, t)$, defined in Eq. (5), as a function of $r \equiv |i - j|$ for a range of sizes between $N = 2^{10}$ and 2^{26} . The data is averaged over about 1000 bond configurations. It is also averaged over times between 2^{14} and half that value. There are clearly strong finite-size effects but the data seems to have converged for the largest sizes, at least up to distances of order 10^4 .

regime (for the short-range case, $d > d_u = 6$), a precise connection can be given between σ and an equivalent d , namely [27–29],

$$d = 2/(2\sigma - 1), \quad (9)$$

and so, for the long-range model, the mean field regime is $1/2 < \sigma < 2/3$.

The connection between critical exponents of the short-range and corresponding long-range models has been discussed systematically in Ref. [27], where it is noted that an exponent of the short-range model in d dimensions is d times the corresponding exponent of the equivalent one-dimensional long-range model. Hence, to get the exponent $\alpha_s = d - 4$ in the static metastate for the long-range model we divide by d and, since we work in the mean field regime, use Eq. (9) to relate d to σ . This gives

$$\alpha_s = 3 - 4\sigma \quad (\text{long-range model}). \quad (10)$$

Here we focus on one value in the mean-field regime, $\sigma = 5/8$, which corresponds to $d = 8$ according to Eq. (9). Using standard finite-size scaling analysis we find that $T_c = 1.85(2)$ for this model with $z_b = 6$. We need to work *well* below T_c so that our data is characteristic of the ordered phase and does not also incorporate critical fluctuations. We take $T = 0.4T_c = 0.74$. We have preferred to focus the numerical effort, which is substantial, on one temperature in order to get the best quality data for the largest possible range of sizes.

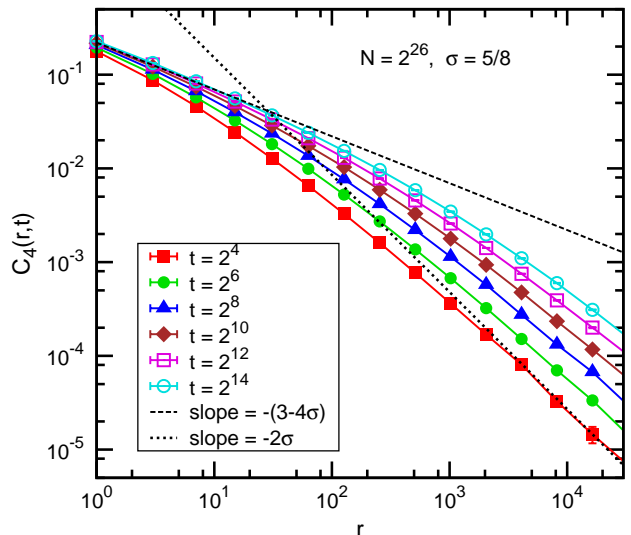


FIG. 2: Data for the correlation function for the largest size $N = 2^{26}$ as a function of r for different times between $t = 2^4$ and 2^{14} . A gradual crossover can be seen between two power laws. At large t and small r , $C_4(r, t) \propto 1/r^{\alpha_d}$ with $\alpha_d = 3 - 4\sigma$ (which is also the value of α_s expected from the static metastate using RSB theory). This is the region in which the data has reached a steady state. At large r and small t one finds $C_4(r, t) \propto 1/r^{2\sigma}$ which is just the average of the square of the interactions J_{ij} .

We quench the system from infinite temperature to $T = 0.74$ at time $t = 0$ and follow the evolution of the system using Monte Carlo simulations. We measure spin correlations, averaging them for times between 2^k and 2^{k+1} , for integer k up to a maximum value. For the largest sizes this was $k = 14$. We find that finite-size effects are very large and we need to study enormously large sizes. We therefore take a range of sizes which also increases geometrically, $N = 2^\ell$ up to $\ell = 26$. We also average over about 1000 samples (the precise number depending on size).

Figure 1 shows our data for the correlation function $C_4(r, t)$, defined in Eq. (5), as a function of $r \equiv |i - j|$ at $t = 2^{14}$ for different sizes. Despite the strong finite-size effects the data seems to have converged for the largest sizes at least for the range of distance presented.

Having established that the largest size, $N = 2^{26}$, is large enough to eliminate finite-size effects for the range of r and t being considered we now discuss the data for this size in more detail. Figure 2 shows data for $C_4(r, t)$ at different times as a function of r . It is expected to have the scaling form shown in Eq. (5). For short range models the scaling function $f(x)$ decays exponentially at large x because the correlation function falls off very rapidly once r is greater than the dynamic correlation length. However, in the present model we have interactions of arbitrarily long range which give a “direct” contribution to the correlation function at large distances. Since $C_4(r, t)$

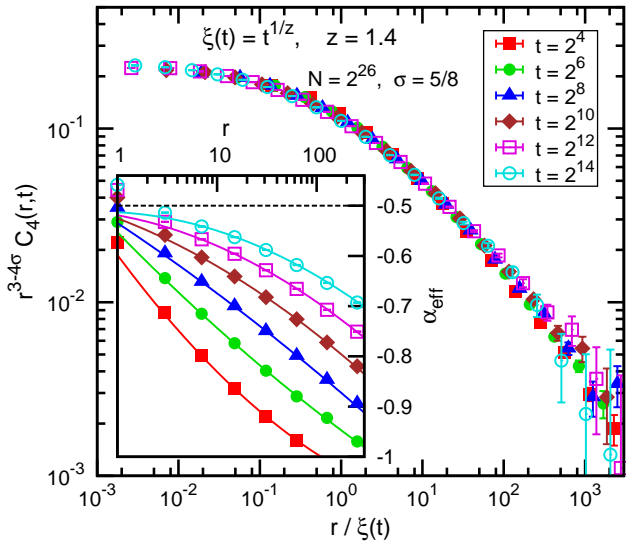


FIG. 3: The main part of the figure is a scaling plot of our data for the largest size $N = 2^{26}$ at $T = 0.74$ according to Eqs. (5) and (3). We assume a pure power law for $\xi(t)$, namely with $\xi(t) = t^{1/z}$. The data collapses well with a dynamic exponent $z(T) = 1.4$. The inset is a plot of the effective exponent α_{eff} , the slope of the curves in Fig. 2, obtained by differentiating a spline fit. The lines are quadratic fits to the data for intermediate r ($7 \leq r \leq 255$). One sees corrections to the parabolic fits at very small distances, $r \leq 3$.

involves the square of the spin-spin correlation function, and is averaged over the interactions, the direct contribution should be proportional to $[J_{ij}^2]$, which, according to Eq. (8), is proportional to $\propto r^{-2\sigma}$ ($= r^{-5/4}$ for $\sigma = 5/8$). Fig. 2 follows this behavior at short times and large distances, see the dotted line.

By contrast, at small r and large t , where $r \ll \xi(t)$, the data for different times collapses and is consistent with a decay $r^{-(3-4\sigma)}$ ($= r^{-1/2}$ for $\sigma = 5/8$), see the dashed line in Fig. 2. To estimate better the slope at large t and small r we plot in the inset to Fig. 3, the effective exponent α_{eff} , the slope of the data in Fig. 2, as a function of r for different times. The curves are quadratic fits for intermediate r ($7 \leq r \leq 255$). The intercepts of the fits approach -0.5 for $r \rightarrow 0$ at large t . Hence, according to Eq. (6), we have $\alpha_d = 3 - 4\sigma$ (or at least very close to it.) However, this is precisely equal to α_s , the corresponding

exponent from the *static* metastate according to RSB theory as shown in Eq. (10). Hence we see that, in the mean field regime, the static and dynamic metastates appear to agree and the description appears to be that of RSB. The latter agrees with some other studies [30], and is also implied by those, such as Refs. [22, 23], which argue that RSB holds even below six dimensions.

The main part of Fig. 3 shows a scaling plot of our data for the largest size according to Eqs. (5) and (3). The data scales well and indicates $z(0.4T_c) = 1.4(2)$. For short-range models $z(T)$ is found to obey Eq. (4). If we assume the same here then $z(T_c) = z_c = 0.56(8)$. To translate this value for z_c , the *critical* dynamical exponent, into the exponent for the equivalent short-range model, we multiply by d ($= 8$), as discussed above, so our estimate for the critical dynamical exponent of the $d = 8$ short-range spin glass is $z_c = 4.5(6)$ ($d = 8$). This model is in the mean field region ($d > 6$) for which the dynamical exponent is found to be $z_c = 4$ [24, 31]. Our result is consistent with this.

To conclude, we have shown quantitatively that the non-equilibrium dynamics following a quench of a model which is a proxy for a short-range spin glass in dimension $d > 6$ is given, in the steady-state regime where the distance is less than the non-equilibrium correlation length, by the *analytic* result for the *static* metastate calculated according to RSB theory. This suggests that (i) RSB theory applies to spin glasses above the upper critical dimension, $d_u = 6$, and (ii) the dynamic and static metastates are equivalent (at least in this region). If the latter is true *in general* it would provide a great simplification in the study of spin glasses.

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